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PARTICLE COLLISION RATE IN TURBULENT FLOW

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Abstract—The fluctuating relative motion of two solid particles or liquid drops in a turbulent gas flow is analysed, yielding an expression for collision rate as a function of the particle concentration and relaxation times and the turbulence intensity and scale. Unlike earlier theories, particles of intermediate size are included, having approach velocities neither well-correlated nor completely independent.

1. INTRODUCTION

Collisions between particles are likely to have a strong influence on the behaviour of a concentrated aerosol, especially where they result in agglomeration. In many situations, turbulence in the suspending gas will be the primary mechanism for relative motion between particles, for example in pneumatic transport or in gas cleaning equipment. Lack of a comprehensive theory for collisions in turbulent flow, and the difficulty of conducting reliable experiments, has meant that the phenomenon tends to be neglected, unlike deposition.

While the equations for the development in time of the size distribution of an agglomerating aerosol (Muller 1928; Drake 1972) may be solved numerically (e.g. Williams & Crane 1979), given the collision rates between any two sizes of particle and the probability of coalescence, evaluation of the collision rates has not been attempted for all relevant combinations of particle size and turbulence parameters. Collision rates are expressed by a collision coefficient C_{12} (dimensions L^3T^{-1}) defined such that $C_{12}N_1N_2$ is the number of collisions in unit time between two sizes of particle having number concentrations N_1 and N_2 (per unit volume).

If the particles are sufficiently small to follow the turbulent motion exactly and be entrained completely by the smallest, energy-dissipating eddies (characterised by the Kolmogorov microscale), the collision rate, determined from the relative motion of two adjacent particles, is a function only of the local fluid velocity gradient. A collision coefficient C_{12} for this turbulent shear collision mechanism was first evaluated by Camp & Stein (1943), by analogy with Smoluchowski's (1917) pioneering work in laminar flows. East & Marshall (1954) introduced the concept of turbulent accelerative coagulation, considering the particles to be entrained in a "parcel" of fluid undergoing turbulent motion as a solid body; collisions then arise from the differing inertial response of unequal particles to the fluid motion.

Saffman & Turner (1956), in a more rigorous analysis of raindrop growth, combined these two mechanisms and additionally accounted for gravitational effects, but still required the particle size to be less than the microscale and the particle relaxation time to be less than the characteristic timescale of the dissipating eddies.

Their result, for two particles denoted by subscripts 1 and 2, is

$$C_{12} = (\pi/2)^{1/2}(d_1 + d_2)^2[(1 - (\rho_G/\rho_P))^2(\tau_1 - \tau_2)^2 \times 1.3(\epsilon^3/\nu_G)^{1/2} + (1/36)(d_1 + d_2)^2(\epsilon/\nu_G)]^{1/2} \quad [1]$$

where the first and second terms on the r.h.s. represent the accelerative and shear mechanisms

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respectively. τ is the relaxation time of a particle of diameter d and material density ρ_P; ϵ is the turbulent energy dissipation rate in the gas, of density ρ_G and kinematic viscosity ν_G.

Considering the shear mechanism only, Delichatsios & Probstein (1974) used simple mean-free-path concepts to obtain, for a monodispersion of particles smaller than the Kolmogorov microscale,

$$C_{12} = 0.40\, d^3\, (\epsilon/\nu_G)^{1/2}. \qquad [2]$$

For this case,[1] reduces to a similar form, but with a constant of 1.67 (or 1.30 from an alternative derivation) instead of 0.40. (Although there appear to be cases of authors misquoting each other's results, there is general agreement that this constant is of order unity; the treatment by Levich (1962), giving a constant an order of magnitude greater, is believed to be physically doubtful.)

With higher energy dissipation rates (10^2–10^4 W/kg, typical of industrial processes rather than atmospheric turblence) and consequently smaller and more vigorous dissipating eddies, approaching particles may no longer have well-correlated velocities; the larger particles, responding imperfectly even to the larger eddies, will tend to approach each other from separate eddies with random and independent velocities. Abrahamson (1975) evaluated a collision coefficient for this opposite extreme case, obtaing

$$C_{12} = 1.25(d_1 + d_2)^2(v_1'^2 + v_2'^2)^{1/2} \qquad [3]$$

where v' is the r.m.s. particle velocity, related to r.m.s. fluid velocity u' by

$$v' = u'(1 + 1.5\, \tau\epsilon/u'^2)^{-1/2}. \qquad [3]$$

Equation [3] was estimated to be valid when

$$d^2 > 15\nu_G u'^2(\rho_G/\rho_P)/\epsilon. \qquad [5]$$

Few reliable experimental studies of turbulent agglomeration in pipes have been reported, and these are largely confined to the shear collision regime. Yoder & Silverman (1967) measured number concentrations in a near-mondisperse aerosol of sub-micron particles and deduced separate agglomeration and deposition coefficients, assuming them to be independent of particle size. A hydrosol of similar particles, again smaller than the turbulent microscale, was sampled by Delichatsios & Probstein (1974); analysis of the size distributions yielded a mean agglomeration coefficient in good agreement with theory (making use of a polydispersity factor). Attempts to repeat the experiment for particles larger than the microscale were unsuccessful because of break-up of agglomerates. Okuyama *et al.* (1978) adopted a similar approach, comparing a solution of the equation for evolution of the size distribution with data from a flow of submicron droplets in air. They found broad agreement with shear collision theory for smaller particles, but noted increased agglomeration with larger particles which was not inconsistent with a prediction using [1].

Lack of a theory or any relevant data for particles of intermediate size, which have partially-correlated velocities before collision, prompted the work described in this paper. The aim was to derive an expression for C_{12} for the accelerative collision mechanism, spanning the whole particle size range between the extremes represented by [1] and [3], and with particular reference to pipe flows. (Other collision mechanisms, resulting from Brownian motion, gravity, sound waves, etc., are not treated here.)

2. PARTICLE MOTION IN TURBULENT FLOW

The relative velocity between two adjacent particles is clearly a key determinant of the particle collision rate. Evaluation of the relative velocity of intermediate size particles is complicated by their unknown degree of velocity correlation. A relative velocity between two adjacent particles can be attributed to three effects: (i) existence of a fluid velocity gradient between the two particles (separation dependent); (ii) different inertial response of unequal particles to a common entraining eddy (separation independent); (iii) projection of large particles towards each other from uncorrelated parts of the flow (separation independent).

Staffman & Turner (1956) show that effect (i) is negligible at particle separation r satisfying.

$$r/R \ll 0.38\rho_P(\nu_G\epsilon)^{1/4}(d_1 - d_2)/\mu_g$$

where the collision radius $R = \frac{1}{2}(d_1 + d_2)$ and μ_G is the dynamic viscosity of the gas. For a pipe flow, it is therefore usually valid to assume that relative velocities are independent of separation for $r/R \sim 0\{1\}$; the error introduced for particles of equal size is unlikely to have a large effect on the development of a typical polydisperse system. If this condition is satisfied, then relative velocity can be evaluated at a hypothetical zero separation, and applied at small separation. It is therefore proposed to evaluate that relative velocity by extending the solution for the velocity of a single particle. The resulting value is then used in a kinetic collision model, and hence the collision rate is obtained explicitly. Discussing viscous interaction effects immediately prior to the collision of two deformable particles, Delichatsios (1980) suggests several dimensionless groups on which collision rate may depend, including the effect of the fluctuating pressure difference between opposite sides of the particles. However, such effects are neglected here, owing to the lack of any suitable experimental or analytical work on which to base the treatment. (As a first approximation, the expression for collision rate to be derived in this paper might be multiplied by an empirically-obtained collision efficiency.)

2.1 *Motion of a single particle*

The motion of a spherical particle in a turbulent gas is described approximately, in standard tensor notation, by

$$\dot{v}_i + v_i/\tau = b\dot{u}_i + u_i/\tau + \frac{2}{3}b\left[(u_j - v_j)\frac{\partial u_i}{\partial x_j} - \nu\nabla^2 u\right] - \frac{2}{3}b[(\rho_P/\rho_G) - 1]\,g\delta_{ig} \quad [6]$$

where u_i and v_i are the fluctuating parts of the gas and particle velocities, respectively, in the i-direction, x being the general coordinate; $b = 3\rho_G/(2\rho_P + \rho_G)$, g is the gravitational acceleration, and δ_{ig} is the Kronecker delta (= 1 when g is in direction i, 0 otherwise). Neglect of the Basset "history" term implies $b \ll 1$ and the absence of high accelerations due to external forces. The validity of [6] is discussed at some length by Williams (1980); the most restrictive condition for its applicability to gas-particle flows is that the drag force on a particle is given by Strokes' law.

Two well-documented (e.g. Hinze 1975) simplifications of [6] are given by the omission of the last term on the r.h.s., or of the last two terms. These will be denoted here by [6a] and [6b] respectively. Their validity has been examined by a Monte Carlo simulation of particle trajectories in a three-dimensional, random gas velocity field, generated using Kraichnan's (1970) method; details are given by Williams (1980). In figure 1, computed values of the ratio v'/u' of particle and gas r.m.s. fluctuating velocities, averaged over all three directions, are plotted for turbulence parameters roughly representative of fully-developed air flow in a 100 m

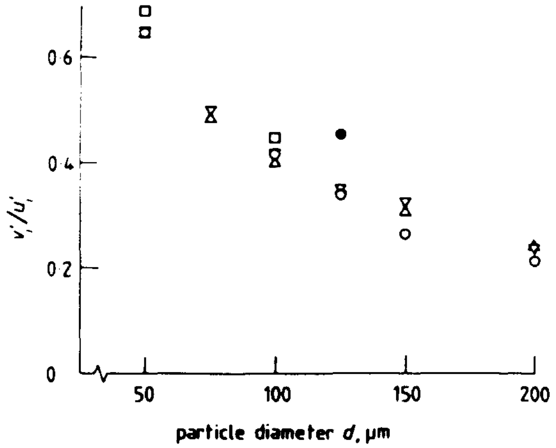

Figure 1. Ratio of particle and gas r.m.s. velocities from Monte Carlo simulations, using [6], ○: [6a], ▽: [6b], △: and [6], with non-Stokesian drag law ●. Reeks (1977) □.

dia. ripe at Reynolds number 10^5, with $\rho_P = 1000\,\text{kg/m}^3$. Two values estimated from Reeks' (1977) results are also shown for comparison. The agreement between results from [6], [6a] and [6b] indicated that the equation

$$\dot{v}_i + v_i/\tau = b\dot{u}_i + u_i/\tau \qquad [6b]$$

is a valid approximation to [6] for $d < 200\,\mu\text{m}$ under these conditions. This enables relationships between gas and particle energy spectra, obtained by various authors on the basis of [6b], to be employed in the subsequent analysis. It also allows the gravitational coalescence mechanism, if this is important, to be treated separately from the turbulence mechanism which is the subject of this paper. It is expected that collisions caused by gravity may be neglected if $V_T \ll v'$ where V_T is a particle terminal velocity; this condition becomes $d \ll 200\,\mu\text{m}$ approximately, for the pipe flow referred to above.

An indication of the applicability of any equation of motion including Stokes drag was given by a further Monte Carlo simulation, using an equation derived from Tchen (1947) equation in the same way as [6] but with particle drag coefficient C_D expressed by Ingebo's (1956) empirical relation (used for its convenient form rather than its accuracy). The point plotted for $d = 125\,\mu\text{m}$ illustrates the magnitude of the error introduced by the Stokes drag assumption. For these particular flow conditions, if particles with diameters above 50–100 μm form only a small fraction of the total number, it is considered reasonable to use a collision model which necessarily relies on [6b].

A statistical description of the gas and particle motions is provided by their respective Lagrangian energy spectrum functions, $E_{GL,ii}\{\omega\}$ and $E_{PL,ii}\{\omega\}$, where ω is the angular frequency. Where [6b] is valid, it is not difficult to show, e.g. Hinze (1975), that the fluid and particle energy spectra are related by

$$\begin{aligned} E_{PL,ii}\{\omega\}/E_{GL,ii}\{\omega\} &= [1 + b^2(\omega\tau)^2]/[1 + (\omega\tau)^2] \\ &\approx [1 + (\omega\tau)^2]^{-1} \text{ for } \rho_P \gg \rho_G. \end{aligned} \qquad [7]$$

Unfortunately, [7] is not immediately useful since it is difficult to measure the Lagrangian spectrum of the gas in a real flow. However, evidence exists (Weinstock 1976) which suggests that the Lagrangian and Eulerian "moving with the mean flow" autocorrelation functions,

$R_L\{\phi\}$ and $R_E^*\{\phi\}$ respectively, are quite similar; that is,

$$R_L\{\phi\} \approx R_E^*\{\phi\} \approx R_E\{\phi u_i'/\bar{u}\}$$

where

$$R_E\{\phi\} = \overline{(u_i\{t\}u_i\{t+\phi\})}/u_i'^2.$$

$$= \int_0^\infty E_{Eii}\{\omega\}\cos\omega\phi \, \mathrm{d}\omega/u_i'^2.$$

Time and time delay are denoted by t and ϕ respectively, and $\bar{u}$ is the local mean gas velocity. The corresponding integral time scales T, defined by $T = \int_0^\infty R\{\phi\}\,\mathrm{d}\phi$, are related by

$$T_L \approx T_E^* = T_E\bar{u}/u_i' = L_f/u_i'$$

where L_f is the longitudinal integral space scale.

Recalling that the autocorrelation function and 1-D Eulerian energy spectrum are cosine Fourier tansform pairs, it can be seen that

$$E_{GLii}\{\omega\} \approx (\bar{u}/u_i')E_{GEii}\{\omega\bar{u}/u_i'\}.$$

The non-dimensional wavenumber spectrum $F_{Gii}(ak_i)$ is now introduced, referred to an observer moving with the mean velocity $\bar{u}$. Referring to a pipe flow, it is defined by

$$F_{Gii}\{ak_i\} = E_{Gii}\{k_i\}/(au_i'^2) = \bar{u}E_{GEii}\{\omega\}/(au_i'^2)$$

where a is the pipe radius, and

$$\int_0^\infty F_{Gii}\{ak_i\}\,\mathrm{d}(ak_i) \equiv 1.$$

It can be shown (Williams 1980) that

$$E_{GLii}\{\omega\} \approx au_i'F_{Gii}\{ak_i\}$$

where $\omega = u_i'k_i$. Hence [7] becomes

$$E_{PLii}\{\omega\} = [au_i'/(1+\omega^2\tau^2)]F_{Gii}\{ak_i\}. \qquad [8]$$

The 1-D wavenumber spectrum of the gas can be approximated by various empirical forms, of which two will be considered here. It is assumed that the presence of particles does not appreciably change the shape of the spectra. Damping of the high-wave number fluctuations by a dispersed phase (with a consequent reduction in u' and ϵ) is a well-known phenomenon (e.g. Al Taweel & Landau 1977), which is likely to become significant when the particle volume fraction exceeds about 10^{-4} in a typical gas-particle flow. This could therefore be a rather restrictive assumption, since collision rates ($C_{12}N_1N_2$) would be expected to be greatest when the particle concentrations N are highest. There is, however, an opposite effect, turbulence augmentation, if the particles are large enough to have a significant gravity-driven mean slip velocity relative to the gas (e.g. Theofanous & Sullivan 1982). As yet, insufficient data are available to permit a better representation of F_{Gii} than expressions for single-phase flows.

The simplest empirical description of the spectrum characterises the turbulent flow by two parameters, the r.m.s. fluctuating velocity u_i' and the longitudinal scale L_f. It is written

$$F_{Gii}\{ak_i\} = (2/\pi)(L_f/a)/[1+(L_f/a)^2(ak_i)^2] \qquad [9]$$

which corresponds to an exponential form of the longitudinal space correlation coefficient of velocity, i.e.

$$f\{\chi_i\} = (\overline{u_i\{x_i\}u_i\{x_i+\chi_i\}})/u_i'^2 = \exp -\{\chi_i/L_f\},$$

where x and χ denote distance and separation respectively.

To model the small scale turbulence, it is necessary to introduce a further parameter, the microscale l_f, defined by

$$l_f = u'(15\nu/\epsilon)^{1/2}.$$

Noting that the space correlation is parabolic for small separation (Hinze 1975), i.e.

$$f\{\chi_i\} = 1-(\chi_i/l_f)^2 \text{ for } \chi_i \ll l_f,$$

it can be shown (Williams 1980) that a more accurate form of the wave-number spectrum is

$$F_{Gii}\{ak_i\} = (2/\pi)(\gamma/[\gamma-1])[(L_f/a)/[1+(L_f/a)^2(ak_i)^2] - (L_f/a)/[\gamma^2+(L_f/a)^2(ak_i)^2]] \qquad [10]$$

where $\gamma = 2(L_f/l_f)^2$.

Now the mean square particle fluctuating velocity is given by

$$v_i'^2 = \overline{v_i^2} = \int_0^\infty E_{PL,ii}\{\omega\}\, d\omega.$$

Introducing the dimensionless particle relaxation time

$$\theta = \tau u_i'/L_f \; (=\tau/T_L) \qquad [11]$$

and substituting from [8] and [9],

$$\overline{v_i^2} = u_i'^2/(1+\theta), \qquad [12]$$

which has the same form as the expression obtained by Levins & Glastonbury (1972). Similarly, [8] and [10] give

$$\overline{v_i^2} = u_i'^2[\gamma/(\gamma-1)][(1+\theta)^{-1} - (1+\gamma\theta)^{-1}/\gamma].$$

Observing that γ is typically of order $10^2 \to 10^3$ (Lawn 1970), it can be seen that the simple form of the wavenumber spectrum, [9], and the more precise form, [10], give almost identical values of particle r.m.s. fluctuating velocity. This behaviour may be expected since particle trajectories are mainly determined by the large-scale (low wavenumber) fluid motion, which is least affected by the presence of the particles. The r.m.s. fluctuating velocity of a single particle will therefore be calculated from [12].

2.2 *Relative motion of two particles*

The relative velocity $w\{t\}$ between two particles with zero separation, at position x at time t, is given by

$$\langle w_i^2\{t\}\rangle = \overline{v_{1i}^2} + \overline{v_{2i}^2} - 2\langle v_{1i}\{\mathbf{x},t\}v_{2i}\{\mathbf{x},t\}\rangle \qquad [13]$$

where $\langle\ \rangle$ denotes the ansemble average, equal to the time-mean average for stationary, homogeneous turbulence. Therefore calculation of w_i' requires the evaluation of the velocity correlation of two coincident particles.

Integrating [6] once gives (Panchev 1971)

$$v_i\{\mathbf{x},t\} = bu_i\{\mathbf{x},t\} + [2(\rho_P - \rho_G)/(2\rho_P + \rho_G)](1/\tau)\int_0^\infty u_i\{\mathbf{x},t-\phi\}\exp\{-\phi/\tau\}\,\mathrm{d}\phi \qquad [14]$$

where $u_i\{\mathbf{x},t\}$ is the velocity of the gas surrounding a particle at position $\mathbf{x}$ at time t. For $\rho_P \gg \rho_G$, the first term on the r.h.s. can be neglected when the particles are not too large (e.g. $d \ll 0(10^3)\mu$m for the pipe flow referred to in connection with figure 1), while the factor in square brackets in the second term tends to unity.

Observing that particle trajectories over distances of the order of their collision radius will be almost straight and substituting a suitable expression for the fluid velocity covariance function, the following equation, derived in the Appendix, is obtained for the particle velocity correlation:

$$\langle v_{1i}\{\mathbf{x},t\}v_{2i}\{\mathbf{x},t\}\rangle = (\tau_1\tau_2)^{-1}\int_0^\infty E_{Gii}\{k_i\}\int_0^\infty\!\!\int R_L\{\psi-\phi\}\exp\{-(\psi/\tau_1)$$

$$-(\phi/\tau_2)\}\cos\{k_i w_i\{t\}(\psi+\phi)/2\}\,\mathrm{d}\psi\,\mathrm{d}\phi\,\mathrm{d}k_i. \qquad [15]$$

2.3 *Solutions for relative velocity*

Small particles. Provided that one of the particles satisfies the condition $\theta \ll 1$, the argument of the cosine term in [15] is small for all non-negligible values of the integrand. Thus, [15] becomes

$$\langle v_{1i}\{\mathbf{x},t\}v_{2i}\{\mathbf{x},t\}\rangle = u_i'^2(\tau_1\tau_2)^{-1}\int_0^\infty\!\!\int R_L\{\psi-\phi\}\exp\{-(\psi/\tau_1)-(\phi/\tau_2)\}\,\mathrm{d}\psi\,\mathrm{d}\phi.$$

Substituting a form of R_L corresponding to the simple wavenumber spectrum [9] gives

$$\langle v_{1i}\{\mathbf{x},t\}v_{2i}\{\mathbf{x},t\}\rangle = u_i'^2[\theta_1+\theta_2+2\theta_1\theta_2]/[(\theta_1+\theta_2)(1+\theta_1)(1+\theta_2)],$$

from which

$$\langle w_i^2\{t\}\rangle/u_i'^2 = (\theta_1-\theta_2)^2/[(\theta_1+\theta_2)(1+\theta_1)(1+\theta_2)]. \qquad [16]$$

Alternatively, substituting a form of R_L corresponding to the more exact wavenumber spectrum [10] gives

$$\langle w_i^2\{t\}\rangle/u_i'^2 = \frac{\gamma}{\gamma-1}\frac{(\theta_1-\theta_2)^2}{\theta_1+\theta_2}\left[\frac{1}{(1+\theta_1)(1+\theta_2)} - \frac{1}{(1+\gamma\theta_1)(1+\gamma\theta_2)}\right]. \qquad [17]$$

Equations [16] and [17] may be compared with the results of Scaffman & Turner (1956) which, for the accelerative mechanism can be written

$$\langle w_i^2\{t\}]/u_i'^2 = 387\,(\theta_1-\theta_2)^2 \qquad [18]$$

using Lawn's (1970) pipe flow data to relate the small and large length scales, for air at atmospheric pressure. This comparison is shown graphically in figure 2; in addition, all equations give $\langle w_i^2\{t\}\rangle \equiv 0$ for $\theta_1 = \theta_2$.

As expected, [16] and [17] are identical for not-too-small particles, but diverge appreciably for very small particles. The good agreement between [17] and [18] for very small particles is particularly encouraging; bearing in mind that $\langle w_i^2\{t\}\rangle$ cannot exceed $2u_i'^2$, the validity of the Saffman and Turner result for not-too-small particles is dubious.

The predicted behaviour for very dissimilar particles (large values of θ_1/θ_2) is also as expected. Clearly, a large particle has a small fluctuating component of velocity, whereas the fluctuating velocity of a small particle is roughly u_i'. Their mean square relative velocity is therefore approximately $u_i'^2$, which is indeed exhibited by [16] and [17].

Large particles. When one of the particles satisfies the condition $\theta \gg 1$, the integrand of [15] is negligible unless $\phi \approx \psi$. Substituting the simple form of the wavenumber spectrum and the corresponding form of R_L, [15] becomes

$$\langle v_{1i}\{\mathbf{x},t\}v_{2i}\{\mathbf{x},t\}\rangle = (\tau_1\tau_2)^{-1}\int_0^\infty (2/\pi)u_i'^2 L_f/(1+k_i^2L_f^2)$$

$$\times\int_0^\infty\int \exp\{-\phi(\tau_1^{-1}+\tau_2^{-1})-|\psi-\phi|/T_L\}\cos\{k_iw_i\{t\}\phi\}\,d\psi\,d\phi\,dk_i.$$

Substituting the r.m.s. value of $w_i\{t\}$ assuming uncorrelated velocities in the cosine argument, it can be shown (Williams 1980) that

$$\langle w_i^2\{t\}\rangle/u_i'^2 = (1+\theta_1)^{-1}+(1+\theta_2)^{-1}-4[\theta_1+\theta_2+\theta_1\theta_2\{(1+\theta_1)^{-1}+(1+\theta_2)^{-1}\}^{1/2}]^{-1}. \qquad [19]$$

It is pertinent to compare [19] with the relative velocity assuming independent approach velocities (Abrahamson 1975). This comparison is made in figure 3. As expected, the curves are identical for very large particles, but diverge for not-too-large particles.

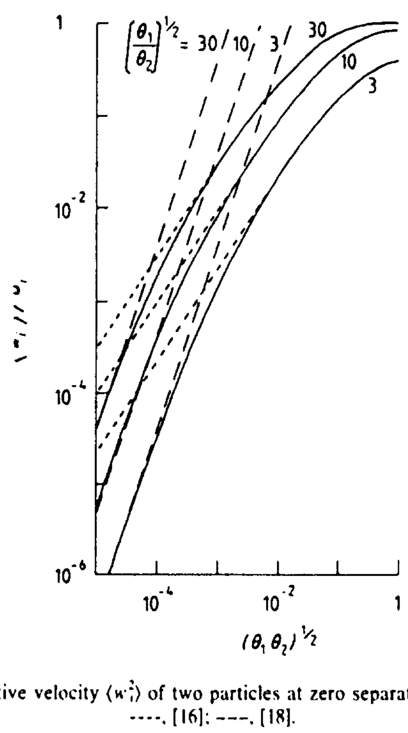

Figure 2. Mean square relative velocity $\langle w_i^2\rangle$ of two particles at zero separation when θ_1 or $\theta_2 \ll 1$. –, [17]; ····, [16]; ---, [18].

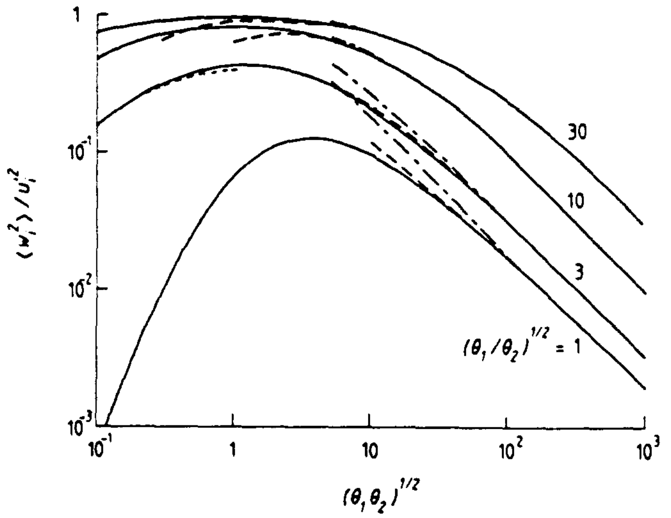

Figure 3. Mean square relative velocity $\langle w_i^2 \rangle$ of two particles at zero separation. –, [20]; ---, [19] (where different from –); ----, [16] (where different from –). ----, Independent appraoch velocities.

Universal solution. Equation [16], for small particles (assuming the simple form of the wavenumber spectrum), is also plotted on figure 3. It can be seen that, for unequal particles, the sets of curves for small and large particles show good compatability in the region $\theta = 0(1)$. It is therefore reasonable to seek a universal solution applicable over all θ, which reduces closely to [16] and [19] for small and large particle respectively. Such a solution is

$$\langle w_i^2\{t\}\rangle / u_i'^2 \approx \{(\theta_1+\theta_2)^2 - 4\theta_1\theta_2[(1+\theta_1+\theta_2)(1+\theta_1)^{-1}(1+\theta_2)^{-1}]^{1/2}\}/\{(\theta_1+\theta_2)(1+\theta_1)(1+\theta_2)\} \quad [20]$$

which is also plotted in figure 3. It is seen to coincide very closely with [16] and [19] over the whole range of $(\theta_1\theta_2)^{1/2}$ and to provide the desired relationship for the region $(\theta_1\theta_2)^{1/2} = 0(1)$.

3. PARTICLE COLLISIONS

The relative velocity between two particles has been obtained above, for small separation of the order of the collision radius R. This result is now used to calculate the particle collision rate. The collision process has been divided into two separate phases; the rate of approach is first calculated using diffusion theory, then the outcome of the collision event is evaluated using a kinetic model.

The reason for this division is that a non-zero approach flux of particles relative to a "target" particle requires a non-zero concentration gradient. This gradient will be appreciable for distances from the "target" not much greater than R. Clearly, this distinction is irrelevant for large particles, whose trajectories are determined at large separation. However, the effect of non-uniform concentration at small separations is appreciable for two small particles.

3.1 *Diffusional approach*

Two particular, but arbitrarily chosen, particles are considered. It is required to calculate the probability of these particles approaching to a separation $< r_0$ in unit time. The value of r_0 must be small enough that the particles' relative velocity at separation r_0 does not differ appreciably from its value at hypothetical zero separation, but sufficiently large that the particle concentration is uniform for $r > r_0$. These are clearly conflicting requirements, but values of r_0 in the range $3R$–$5R$ represent a reasonable compromise.

Now, if the particles have separation $r(> r_0)$ at time t, their separation will be less than r_0 at time $t+\delta t$ if their relative velocity along the line of centres, w_r, exceeds $(r-r_0)/\delta t$. The probability of this is

$$\int_{(r-r_0)/\delta t}^{\infty} \psi\{w_r\}\, \mathrm{d}w_r$$

where $\psi\{w_r\}$ is the probability density of relative velocity w_r. Assuming that this is Gaussian, with variance $w_r'^2$, so that

$$\psi\{w_r\} = [(2\pi)^{1/2} w_r']^{-1} \exp\{-\tfrac{1}{2} w_r^2 / w_r'^2\},$$

the probability of the particles approaching to within r_0 in unit time is given by

$$\lim_{\delta t \to 0} \left[\frac{1}{\delta t} \int_{r_0}^{\infty} 4\pi r^2 \int_{(r-r_0)/\delta t}^{\infty} \psi\{w_r\}\, \mathrm{d}w_r\, \mathrm{d}r \right] = (8\pi)^{1/2} r_0^2 w_r'.$$

Hence the flux Q of particles to within r_0 of a "target" particle is

$$Q = (8\pi)^{1/2} r_0^2 w_r' N. \qquad [21]$$

The same result was obtained, from different reasoning, by Abrahamson (1975).

It is further assumed that w_r is isotropic and that the r.m.s. relative velocity w_r' along the line of centres equals that in an arbitrary direction, w_i'. Saffman & Turner (1956) have shown that this anomaly introduces only a small error into the proportionality constant for the collision rate. Allowance for anisotropy of the relative motion is beyond the scope of the present work, so the possibility of significant error close to the wall in a pipe flow, for example, must be borne in mind.

3.2 *Kinetic collision model*

It is required to calculate the collision probability of a pair of particles which have approached to within a separation r_0 by diffusion. An adequate model for the subsequent particle motion is suggested by observing that the velocity of a particle at time t is almost independent of its velocity at time t_0 if $t-t_0 \gg \tau$, while its velocity is roughly constant over an interval of time $\ll \tau$. Denoting one particle as the target and the other as the projectile, this model makes the assumption that the trajectory of the projectile relative to the target can be approximated by straightline segments of time duration $T_r = (\tau_1\tau_2)^{1/2}$, and that consecutive trajectory segments are independent. In time T_r, the r.m.s. displacement of the projectile relative to the target is $T_r w_i'$, so the average length of each trajectory segment is $s = 3^{1/2} w_i' T_r$ which can be written in dimensionless form as

$$s/R = (2/3)^{1/2} [(\rho_P/\rho_G)(u_i' L_f/\nu_G)]^{1/2} [(\theta_1\theta_2)^{1/2}/(\theta_1^{1/2} + \theta^{1/2})] w_i'/u_i', \qquad [22]$$

with reference to [11].

The probability density of the particles' separation r' at time t will be denoted by $p\{r', t\}$. During the interval $\{t, t+T_r\}$, one of three things will happen: either the particles will collide, or their separation will increase to more than r_0, or their separation will remain less than r_0. Williams (1980) has shown that the probability of collision during $\{t, t+T_r\}$ is

$$\tfrac{1}{2}[1 - \{1 - (R/r')^2\}^{1/2}] \text{ for } r' < (R^2 + s^2)^{1/2},$$

$$[R^2 - (r'-s)^2]/(4r's) \text{ for } (R^2+s^2)^{1/2} < r' < r + s$$

and

$$0 \text{ for } r' > R + s.$$

Also, the probability density of separation at time $t + T_r$, given separation r' at time t, is given by

$$p\{r, t + T_r; r', t\} = r/(2r's) \text{ for } |s - r'| < r < s + r',$$

from which

$$p\{r, t + T_r\} = \int_0^\infty p\{r, t + T_r; r', t\}\, p\{r', t\}\, \mathrm{d}r'.$$

Hence, the probability of separation exceeding r_0 at time $t + T_r$ is

$$\int_{r_0}^\infty p\{r, t + T_r\}\, \mathrm{d}r.$$

It is now possible to evaluate probabilities of the different outcomes at successive instants using a time-marching method on a computer. This gives the collision probability of an arbitrary pair of particles which approach to within r_0 of each other, $P\{r_0\}$. These computations have been performed for a range of values of s and for $r_0 = 3R$ and $3R$. Noting from [21] that the rate of diffusional approach to separation r_0 is proportional to r_0^2, it is expected that $P\{r_0\}$ should vary inversely with r_0^2 since the final collision rate should be independent of r_0. Accordingly, results of the kinetic model are plotted in the form $(r_0/R)^2 P\{r_0\}$ in figure 4.

It can be seen that results for the two values of r_0 agree reasonably well for not-too-small values of s/R (i.e. for all but the smallest particles). Also, for large s/R, the collision probability approaches the value assuming uniform concentration, as used by Saffman & Turner (1956). For computational convenience, a simple empirical formula for $P\{r_0\}$ has been fitted to the results; this is

$$P\{r_0\} = (2/\pi)(R/r_0)^2 \tan^{-1}[\tfrac{1}{2}(s/R)^2]. \qquad [23]$$

3.3 *Complete collision model*

The probability $Q\{r_0\}$ of two particular, arbitrarily chosen particles approaching to within a separation r_0 in unit time has been derived in section 3.1. In section 3.2, the probability $P\{r_0\}$ of

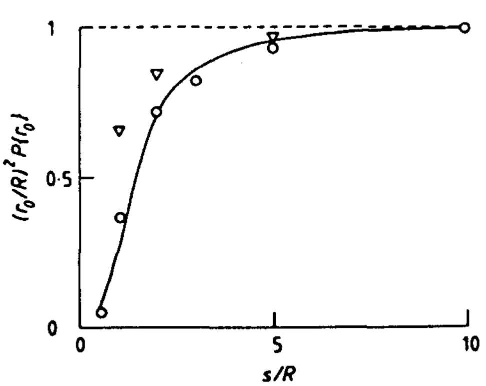

Figure 4. Collision probability $P\{r_0\}$ of two particles with separation r_0 as a function of relative inertial path length s. Computed: r_0/R = 3∇, 5O. –, [23]; ----, without kinetic model.

two such particles colliding has been evaluated. The collision coefficient is then given by

$$C_{12} = Q\{r_0\}P\{r_0\}/N$$

$$\approx [(8\pi)^{1/2}R^2 w_i'][(r_0/R)^2 P\{r_0\}], \quad [24]$$

using eqn. [21] and [23]. When $(r_0/R)^2P\{r_0\} \rightarrow 1$, i.e. when the diffusional collision model alone is applicable, [24] has the same form as [3], but with w' evaluated differently. In terms of dimensionless relaxation time θ, [24] may be written

$$C_{12} \approx [(8\pi)^{1/2}(18/4)\nu_G L_f(\rho_G/\rho_P)(w_i'/u_i')(\theta_1^{1/2} + \theta_2^{1/2})^2]$$

$$\times [(2/\pi)\tan^{-1}\{(1/3)(\rho_P/\rho_G)(u_i'L_f/\nu_G)(w_i'/u_i')^2\theta_1\theta_2/(\theta_1^{1/2} + \theta_2^{1/2})^2\}] \quad [25]$$

with (w_i'/u_i') given as a funciton of θ by [20]. Figure 5 illustrates the effect of introducing the kinetic collision model, showing that the predicted collision rate between particles of equal size falls sharply below that calculated from the diffusion model alone when $(\theta_1\theta_2)^{1/2}$ falls below about 10^{-1}. For unequal particles, as the difference in size increases, this reduction in collision rate becomes operative at decreasing values of $(\theta_1\theta_2)^{1/2}$.

In figure 6, C_{12} is plotted for various particle size combinations for a particular pipe flow. Also shown, withing their ranges of validity, are the collision coefficients calculated from the theories of Saffman & Turner (1956) and Abrahamson (1975), which are somewhat higher than those predicted by the present theory. This results from the entirely diffusional approach of Saffman & Turner, equivalent to $(r_0/R)^2P\{r_0\} \equiv 1$ for all sizes, and from Abrahamson's use of the parameters of the small-scale turbulence to calculate particle fluctuating velocities. It should be remembered that both the present theory and that of Abrahamson make use of results derived from the Tchen equation with the assumption of Stokes' drag law, so neither is strictly valid for $d > 100\ \mu$m approximately, the lower limit [5] for Abrahamson's independent velocity model. In this range, Figure 6 serves merely to compare the two theories.

The predicted influence of turbulence intensity and scale may be judged from the slopes of the curves in figure 6; for example, a doubling of turbulence intensity, which doubles the value of θ for a given drop size and turbulence scale, can increase the collision rate by an order of magnitude. In practice, estimation of the appropriate values of u' could be subject to considerable uncertainty; only recently have unbiased measurement techniques been developed for the continuous phase in the presence of the dispersed phase (e.g. Lourenço & Riethmuller 1982). However, a simple correction to the single-phase value might be employed, such as that

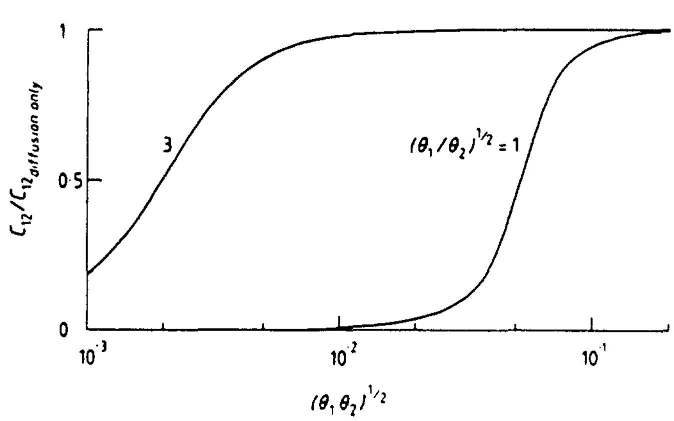

Figure 5. Ratio of collision coefficients C_{12} with and without inclusion of kinetic model.

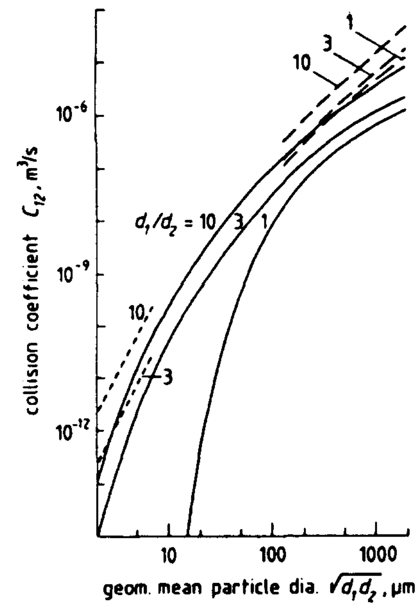

Figure 6. Collision coefficient C_{12} between particles of diameter d_1 and d_2 for pipe flow with $u'_r = 1$ m/s, $L_f = 50$ mm, $\rho_P/\rho_G = 850$, $\nu_G = 15 \times 10^{-6}$ m²/s (e.g. water drops in air in a 100 mm dia. pipe at Reynolds number 10^5). –, [25]; ----, Saffman & Turner (1956); –-, Abrahamson (1975).

developed by Laats and Frishman, quoted by Delichatsios (1980) which depends only on ρ_P/ρ_G and particle volume fraction.

Use of the present theory for prediction of particle or droplet coalescence is, of course, dependent on knowledge of the outcome of each collision (coalescence, perhaps followed by break-up, or bouncing). Prediction of coalescence efficiency is beyond the scope of this paper; however, assuming a value of unity, Crane & Williams (1981) have incorporated the collision model into a two-dimensional numerical method for predicting the evolution of a droplet size spectrum in a turbulent pipe flow. A turbulent deposition model (Williams & Crane 1981) was also included. In the absence of any suitable data, an experiment was carried out to provide a test case (Ow 1980; Williams 1980). Qualitative agreement with experiment was obtained (Crane & Williams 1981), but the extent of uncertainties in various parameters required by the model, and of the experimental errors, means that improved measurement techniques will be required for a more precise verification of the theory. (A notable experimental difficulty is in obtaining sufficiently high particle concentrations N for the rate of coalescence, roughly proportional to N^2, to exceed the rate of turbulent deposition, roughly proportional to N, by an amount sufficient for errors in prediction and measurement of deposition not to mask the effects of coalescence).

4. CONCLUSIONS

By relating the fluctuating velocity of a particle to that of the gas and hence evaluating the fluctuating relative velocity between two given particles, the probability of close approach of the particles in a given time has been calculated, in terms of the particle concentration, particle relaxation times, turbulence intensity and scale. A kinetic collision model then yields the probability of collision.

The resulting analytical expression for collision rate spans the intermediate size range between small particles, having well-coorelated approach velocities, and large particles whose approach velocities are independent. At these two extremes, the present collision model represents an advance on earlier theories by allowing for non-uniform concentration of small particles at small separations and, for large particles, being based on the parameters of the large-scale turbulent motion rather than those of the smallest, energy-dissipating eddies.

Further development of the model, to enhance its applicability to larger particles, will include the addition of gravitational coalescence (or the equivalent mechanism in a centrifugal force field). Removal of the Stokesian drag restriction will present a greater problem and could require a considerable amount of fundamental work on the relationship between gas and particle energy spectra. For small particles, inclusion of the turbulent shear mechanism in the expression for C_{12} would be worthwhile.

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APPENDIX

Evaluation of velocity correlation term in [12]

Using [14] with $\rho_P \gg \rho_G$, the velocity correlation of two coincident particles is

$$\langle v_{1i}\{\mathbf{x}, t\}\, v_{2i}\{\mathbf{x}, t\}\rangle = (\tau_1\tau_2)^{-1} \int_0^\infty \int \langle u_{1i}\{t-\psi; \mathbf{x}, t\} u_{2i}\{t-\phi; \mathbf{x}, t\}\rangle$$

$$\exp\{-(\psi/\tau_1)-(\phi/\tau_2)\}\, \mathrm{d}\psi \mathrm{d}\phi \qquad [A1]$$

where $u\{t'; \mathbf{x}, t\}$ denotes the velocity at time t' of the fluid surrounding the particle which will be at $\mathbf{x}$ at time t. The integrand of [A1] becomes small for $\psi \gg \tau_1$ or $\phi \gg \tau_2$. Since the velocity of a particle cannot change appreciably over times much less than its relaxation time τ, the fluid velocity correlation $\langle u_{1i}u_{2i}\rangle$ may be approximated by

$$\langle u_{1i}\{t'; \mathbf{x}, t\} u_{2i}\{t''; \mathbf{x}, t\}\rangle \approx \langle u_i\{\mathbf{x} + v_{1i}\{\mathbf{x}, t\}(t-t')\mathbf{i}, t'\}\, u_i\{\mathbf{x} + v_{2i}\{\mathbf{x}, t\}(t-t'')\mathbf{i}, t''\}\rangle$$

$$\approx \langle u_i\{\mathbf{x}, t'\} u_i\{\mathbf{x} + w_i\{t\}(t - \tfrac{1}{2}[t'+t''])\mathbf{i}, t''\}\rangle. \qquad [A2]$$

A commonly-used empirical form of this covariance function (e.g. Kraichnan 1970) is

$$\langle u_i\{\mathbf{x}, t'\} u_i\{\mathbf{x} + r\mathbf{i}, t''\}\rangle = R_E^*\{t'-t''\} \int_0^\infty E_{ii}\{k_i\} \cos\{k_i r\}\, \mathrm{d}k_i. \qquad [A3]$$

R_E^* is the longitudinal Eulerian "moving with the stream" time correlation function, which will be set equal to the Lagrangian autocorrelation R_L (see section 2.1).

Substituting [A2] and [A3] into [A1], [15] is obtained. It should be noted again that a hypothetical zero separation between fluid elements is being considered, rather than a separation of $\frac{1}{2}(d_1 + d_2)$ (for two particles in contact); this is consistent with the assumption that the r.m.s. relative velocity of two particles is independent of their separation when the separation is of the same order of magnitude as the collision radius (second paragraph of section 2).